Exercise 11

Evaluate the line integral, where C is the given curve.

$$\int_C xe^{yz}\,ds, \quad C \text{ is the line segment from } (0,0,0) \text{ to } (1,2,3)$$

Solution

The equation of the line going from $(0,0,0)$ to $(1,2,3)$ is

$$\begin{aligned}\mathbf{y} &= \mathbf{m}t + \mathbf{b} \\ &= \langle 1-0, 2-0, 3-0\rangle t + \langle 0,0,0\rangle \\ &= \langle t, 2t, 3t\rangle,\end{aligned}$$

where $0 \leq t \leq 1$. With this parameterization in t, the line integral becomes

$$\begin{aligned}\int_C xe^{yz}\,ds &= \int_0^1 x(t)e^{y(t)z(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_0^1 (t)e^{(2t)(3t)}\sqrt{(1)^2+(2)^2+(3)^2}\,dt \\ &= \sqrt{14}\int_0^1 te^{6t^2}\,dt.\end{aligned}$$

Make the following substitution.

$$\begin{aligned}u &= 6t^2 \\ du &= 12t\,dt \quad \rightarrow \quad \frac{du}{12} = t\,dt\end{aligned}$$

Therefore,

$$\begin{aligned}\int_C xe^{yz}\,ds &= \sqrt{14}\int_{6(0)^2}^{6(1)^2} e^u\left(\frac{du}{12}\right) \\ &= \frac{\sqrt{14}}{12}\int_0^6 e^u\,du \\ &= \frac{\sqrt{14}}{12}(e^u)\Big|_0^6 \\ &= \frac{\sqrt{14}}{12}(e^6 - e^0) \\ &= \frac{\sqrt{14}}{12}(e^6 - 1).\end{aligned}$$